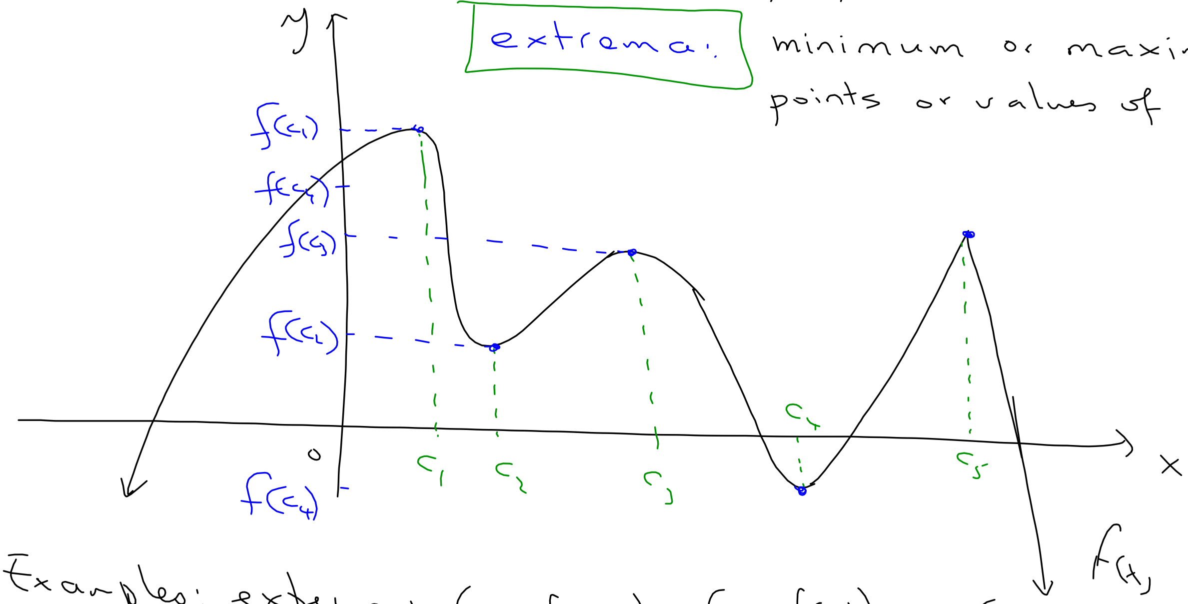
Curve sketching: 1st and 2nd derivatives tests.

- Here, we use the first and 2nd derivatives to analyze the behaviour of functions.

$\boxed{\text{extrema:}}$ minimum or maximum points or values of f



Examples: extrema: $(c_1, f(c_1))$; $(c_2, f(c_2))$; $\ldots$; $(c_5, f(c_5))$

(relative) Minimum extremum: $(c_4, f(c_4))$

(relative) Maximum extremum: $(c_1, f(c_1))$

$\boxed{\text{Critical values:}}$ these are values of $x = c$ in the domain of f satisfying either $f'(c) = 0$ or $f'(c) =$ undefined!

Here the critical values are: $c_1, c_2, \ldots, c_5$

Example: Find all critical values of

$f(x) = 2x^3 - 15x^2 + 24x + 7$ over $(-\infty, +\infty)$

(same as find the values of x where the slope of the tangent line is horizontal or undefined!)

Solution:

$f(x) = 2x^3 - 15x^2 + 24x + 7 \longrightarrow$

$f'(x) = 6x^2 - 30x + 24$

critical values: set $f'(x) = 0 \longrightarrow 6x^2 - 30x + 24 = 0$ Solve for x

$\longrightarrow 6(x^2 - 5x + 4) = 0 \longrightarrow x^2 - 5x + 4 = 0$

$\longrightarrow (x-1)(x-4) = 0 \longrightarrow x = 1,\ x = 4$

Ex: Find the relative max or relative min of

$f(x) = 2x^3 - 15x^2 + 24x + 7$

Since $f'(x) = 6x^2 - 30x + 24 = 0 \longrightarrow$ critical values

$c_1 = 1$ and $c_2 = 4$

We plug each value in f, and compare their outputs

$1 \longrightarrow \boxed{f} \longrightarrow f(1) = 18$ max

$4 \longrightarrow \boxed{f} \longrightarrow f(4) = -9$ min

relative min: $(4, -9)$

relative max: $(1, 18)$

* closed interval a, b

Remark: When given a restricted domain of a function f, say $[a, b]^*$, we must compare the values of f at a and b to those of its critical values as well.

Example: Suppose $f(x) = 2x^3 - 15x^2 + 24x + 7$ has a (restricted) domain $[0, 6]^*$. Find its absolute min and absolute max.

Solutions

	x	$f(x) = 2x^3 - 15x^2 + 24x + 7$ (compute)	
(critical values)	1	$\longrightarrow$	18
	4	$\longrightarrow$	-9 absolute min!
(endpoints)	0	$\longrightarrow$	7
	6	$\longrightarrow$	43 absolute max!

Absolute min: $(4, -9)$

Absolute max: $(6, 43)$

(after comparing all the above output values)

<u>Remark:2</u> The sign of $f'(x)$ helps determine when a function $f(x)$ is increasing or decreasing (monotonicity of f)

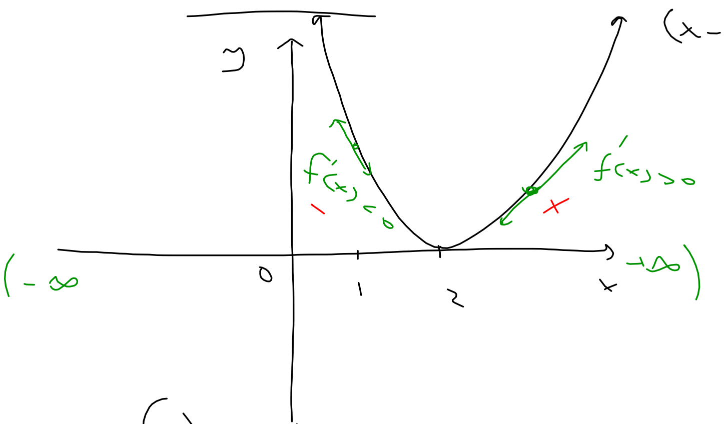
- Suppose f is continuous (without hole or gap) over its domain (a,b) and f is differentiable over (a,b).
- If $f'(x)>0$ for $x\in(a,b) \Longrightarrow f$ is increasing
- If $f'(x)<0$ for $x\in(a,b) \Longrightarrow f$ is decreasing over (a,b)

Ex: $f(x) = (x-2)^2$. Determine the intervals where f is increasing or decreasing

ⓐ from its graph

ⓑ analytically!

Solution



ⓐ Graphically:

Increasing: $(2, +\infty)$

decreasing: $(-\infty, 2)$

ⓑ Analytically

$f(x) = (x-2)^2 \longrightarrow f'(x) = 2(x-2)$

* Sign of $f'(x)$

. set $f'(x) = 0 \longrightarrow 2(x-2) = 0 \longrightarrow x - 2 = 0 \longrightarrow x = 2$

(critical value!)

x:	$-\infty$	2	$+\infty$
sign of $f'(x)$	−		+
f	↘		↗

(Table of signs)

*Test: 0 Test: 3

* Plug "0" for instance into $f'(x)$ and look at the sign of the output value: -4: that is the sign of $f'(x)$

Conclusion:

f is increasing over $(2, +\infty)$

f is decreasing over $(-\infty, 2)$

Remarks: The second derivative helps determine the concavity (bending) of f over its domain

For x in the domain of $f(x)$:

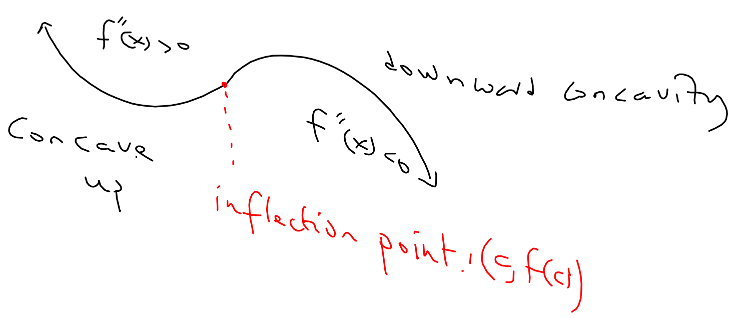
If $f''(x) > 0$, $\Longrightarrow$ $f(x)$ concaves upward

If $f''(x) < 0$ $\Longrightarrow$ $f(x)$ concaves downward

For instance, when $f(x) = (x-2)^2 \Longrightarrow f'(x) = 2(x-2) = 2x-4$

$\Longrightarrow f''(x) = 2 > 0 \longrightarrow$ $f(x)$ concaves upward over its domain $(-\infty, +\infty)$

Note: If $f''(c) = 0$ or undefined and the sign of $f''(x)$ changes around c, the point $(c, f(c))$ is called an inflection point

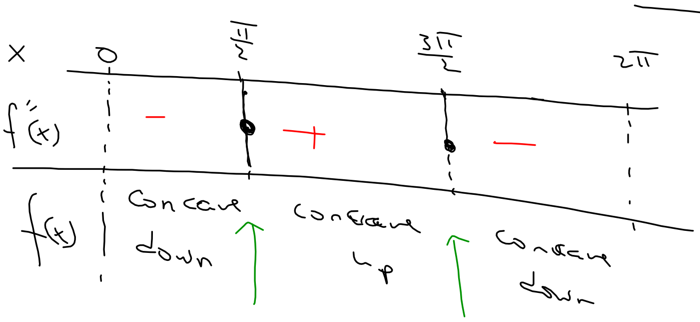


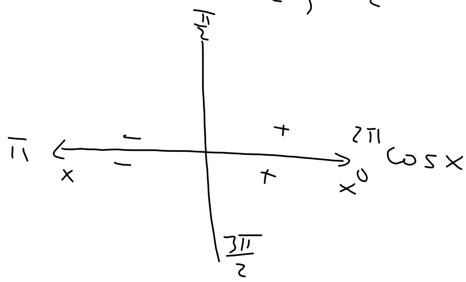
Example: Does $y = \cos x$ admit an inflection point over $(0, 2\pi)$?

$y = \cos x \longrightarrow y' = -\sin x \longrightarrow y'' = -\cos x$

set $y'' = 0 \longrightarrow -\cos x = 0 \longrightarrow \cos x = 0$

$\longrightarrow x = \cos^{-1}(0) = \frac{\pi}{2}, \frac{3\pi}{2}$





$x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ $\Longrightarrow$ $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ are $\boxed{\text{inflection points}}$ Yes!

<u>Example</u>: Let $f(x) = \ln x$, $x > 0$

· $f'(x) = \frac{1}{x}$, is positive since $x > 0$

$\Longrightarrow$ f is increasing over $(0, +\infty)$

Note: $f'(x) \neq 0$, no critical value!

· $f''(x) = -\frac{1}{x^2} < 0$ for all x

$\Longrightarrow$ f concave down over $(0, +\infty)$

Note: although $f''(0) =$ undefined, $0 \notin (0, +\infty)$

So f has <u>No</u> inflection!

See graph!